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THE LIBRATION OF A LUNAR SATELLITE

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BY

DAVID FISHER
THEODORE L. FELSENTREGER

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SUMMARY

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It is shown that the long range variations of the orbital elements of a lunar satellite in libration is periodic, but non-analytic, in nature. A method for finding the period of the motion, as well as the variations of the elements with time, is described.

CONTENTS

THE LIBRATION OF A LUNAR SATELLITE

INTRODUCTION

The basic equations for the long range motion of a lunar satellite lead to two types of motion: circulation and libration. These equations involve the argument of pericenter and the eccentricity (or some function of the eccentricity) as the variables (see Reference 1). When circulation occurs, all values of the argument of pericenter ranging between 0 and 2π, as well as multiples of this range, are permitted. However, when the range of values of g is restricted to a limited set, the motion is called libration. These two types of motion are illustrated by a pendulum which is free to rotate about a horizontal axis. When the pendulum forms complete revolutions about this axis, the motion is circulation; however, if the pendulum oscillates about a vertical axis (e.g., the pendulum of a clock), the motion is called libration.

The equations describing the long period motion of a lunar satellite are similar to the equations for the stellar problem of three bodies discussed by E. W. Brown (Reference 2) over thirty years ago. In an interesting report, Frances Frost (Reference 3) has shown that the method of harmonic analysis used by Brown is well adapted to the calculation of the motion of lunar satellites in the case of circulation. However, this method cannot be applied when libration takes place.

It is possible to determine the argument of pericenter g as a function of its time derivative $\dot{g}$ from the energy equation and the equation for $\dot{g}$ given below. It has been shown in treatises on non-linear mechanics (e.g., see Reference 4) that in the $(g, \dot{g})$ or displacement-velocity space, libration leads to a closed curve or cycle, while circulation leads to an open curve.

It will be shown below that the cycle described by g consists of two distinct branches. This introduces an unexpected asymmetry in the results, leading to a discontinuity in the derivatives of the cycle where the two branches join.

The argument of pericenter g, and consequently $\dot{g}$ and e, can be found as functions of time by integration over these branches. The period of oscillation of the other elements of the lunar satellite may be found by integrating over the entire region of the cycle in the displacement-velocity space.

In order to make the analysis concrete, a specific example is chosen in this report and the calculations necessary to obtain the variations of all the elements as functions of time are given in detail.

ANALYSIS IN THE PHASE SPACE OF A PENDULUM

In order to provide motivation for the analysis of a lunar satellite in the displacement-velocity space (or phase space, as it is often called), we first examine the linear motion of a pendulum. We assume that the angle of oscillation of the pendulum about the vertical is small enough so that the equation of motion can be written in the form

$$\ddot{x} + a^2 x = 0, \tag{1}$$

where x is the angle of displacement from the vertical and $\ddot{x}$ is the second derivative with respect to time.

The solution to Equation (1) can be written in the form

$$\begin{aligned} x &= \pm b \cos (at + c) \\ \dot{x} &= \mp ab \sin (at + c), \end{aligned} \tag{2}$$

where b and c are constants of integration and a is the frequency of oscillation. Equations (2) also satisfy the relation.

$$\frac{x^2}{b^2} + \frac{\dot{x}^2}{a^2 b^2} = 1. \tag{3}$$

Equation (3) is an integral of the motion, and is often called the energy integral.

The analysis in the displacement-velocity space begins with Equation (3). We illustrate the method followed in the more complicated lunar satellite problem by using Equation (3) to derive the results given by Equations (2).

Equation (3) represents an ellipse in the x, $\dot{x}$ space (see Figure 1); the semi-major and semi-minor axes are b and ab, respectively. The motion around the ellipse is taken positive in a counterclockwise direction, and the convention for the four quadrants is taken as shown in Figure 1.

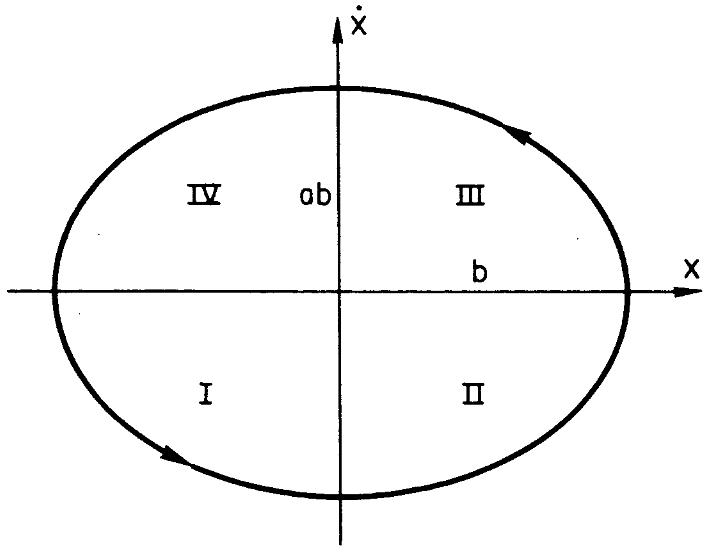

Figure 1

To find the period of oscillation, we note that for the first quadrant,

$$\frac{x}{b} = -\sqrt{1 - \frac{\dot{x}^2}{a^2 b^2}}\ ; \tag{4}$$

consequently,

$$\frac{dx}{\dot{x}} = \frac{1}{a} \frac{d\left(\frac{\dot{x}}{ab}\right)}{\sqrt{1 - \left(\frac{\dot{x}}{ab}\right)^2}} = dt. \tag{5}$$

The period of oscillation T is defined by

$$T = \frac{4}{a} \int_{-1}^{0} \frac{d\left(\frac{\dot{x}}{ab}\right)}{\sqrt{1 - \left(\frac{\dot{x}}{ab}\right)^2}} = \frac{2\pi}{a}. \tag{6}$$

Thus, a represents the frequency of oscillation.

In addition, the indefinite integral from Equation (5) gives

$$\dot{x} = ab \sin (at + c),$$

where c is a constant of integration. Finally,

$$x = -b \cos (at + c),$$

where the negative sign is the result of the sign convention chosen in Equation (4).

This example illustrates the basic principle adopted in this report for the analysis of the libration of a lunar satellite.

THE ENERGY INTEGRAL AND THE EQUATION OF MOTION OF THE ARGUMENT OF PERICENTER

The expressions for the energy integral and the equation of motion of the argument of pericenter may be found in References 5 and 6. The energy equation can be written in the form

$$f_1(\eta) + \eta^3 f_2(\eta) \sin^2 g = 0; \tag{7}$$

the variables are η and g, where $\eta = \sqrt{1 - e^2}$ (e is the eccentricity), and g is the argument of pericenter. The functions $f_1(\eta)$ and $f_2(\eta)$ are defined by

$$\begin{aligned} f_1(\eta) &= -12\frac{\alpha_1}{q}\eta^7 + \left[\frac{\alpha_1}{q}(10 + 6\nu^2) - \frac{4}{3}\frac{C}{nL}\right]\eta^5 - 2\alpha_2\eta^2 + 6\alpha_2\nu^2 \\ f_2(\eta) &= 30\frac{\alpha_1}{q}(\eta^2 - 1)(\eta^2 - \nu^2). \end{aligned} \tag{8}$$

In addition, the equation for the motion of the argument of pericenter is

$$\frac{4}{3}\frac{\dot{g}}{n} = -\left\{\frac{\alpha_1}{q}\,\frac{1}{\eta}\left[-4\eta^2 + 10\left(\eta^2 - \frac{\nu^2}{\eta^2}\right)\sin^2 g\right] + \frac{1 - 5\dfrac{\nu^2}{\eta^2}}{\eta^4}\alpha_2\right\}. \tag{9}$$

The quantities appearing in Equations (8) and (9) are defined in Appendix A.

THE CYCLE IN PHASE SPACE

We have indicated above that libration occurs when the argument of pericenter g is restricted to a finite range of values such that a plot of $\dot{g}$ versus g in the phase space (called a cycle) is a closed curve (e.g., Figure 1).

In order to illustrate such a case of libration, we shall examine Equations (7), (8), and (9) for the initial conditions

$$e = .1, \quad \cos i = .71066905, \quad g = 90°, \quad a = 7.4822577 \text{ moon radii.}$$

For these initial conditions, Equations (8) become

$$f_1(\eta) = -22.947701\,\eta^7 + 22.551912\,\eta^5 - .8609575\,\eta^2 + 1.29143625$$
$$f_2(\eta) = 22.684626\,(2\eta^2 - 1)(\eta^2 - 1). \tag{10}$$

For the convenience of the reader, we list the values for the following constants:

$$\frac{\alpha_1}{q} = 1.9123084 \times 10^{-5}, \qquad \alpha_2 = .43047875 \times 10^{-5}$$

$$\nu^2 = \frac{1}{2}, \qquad n = 3.8097588 \times 10^{-2} \text{ conical (Vanguard) units.}$$

Substitution of Equations (10) into Equation (7) provides a relation between the eccentricity e and the argument of pericenter g. This functional relation is plotted in Figure 2, which indicates a distorted "ellipse." The variable g ranges between the values 75.°96481 and 104.°03519, while e varies between .1 and .30694755. The major axis of this "ellipse" is at e = .17962767, which may be considered a "mean" value of sorts. Adopting the quadrant convention shown in Figure 1, the numerical values of e and 90° − g are listed in Table 1.

The next step is to find $\dot{g}$ as a function of g, which can be readily accomplished by substituting the values of e and 90° − g in Table 1 into Equation (9). It is found that the values of $\dot{g}$ in 2.8573191×10^{-7} canonical units vary from −1.0396376 to 1.0950486—these values are also listed in Table 1.

Figure 3 shows the 90° − g, $\dot{g}$ cycle derived from the values in Table 1. It is seen that the figure departs only slightly from an ellipse, with the branch

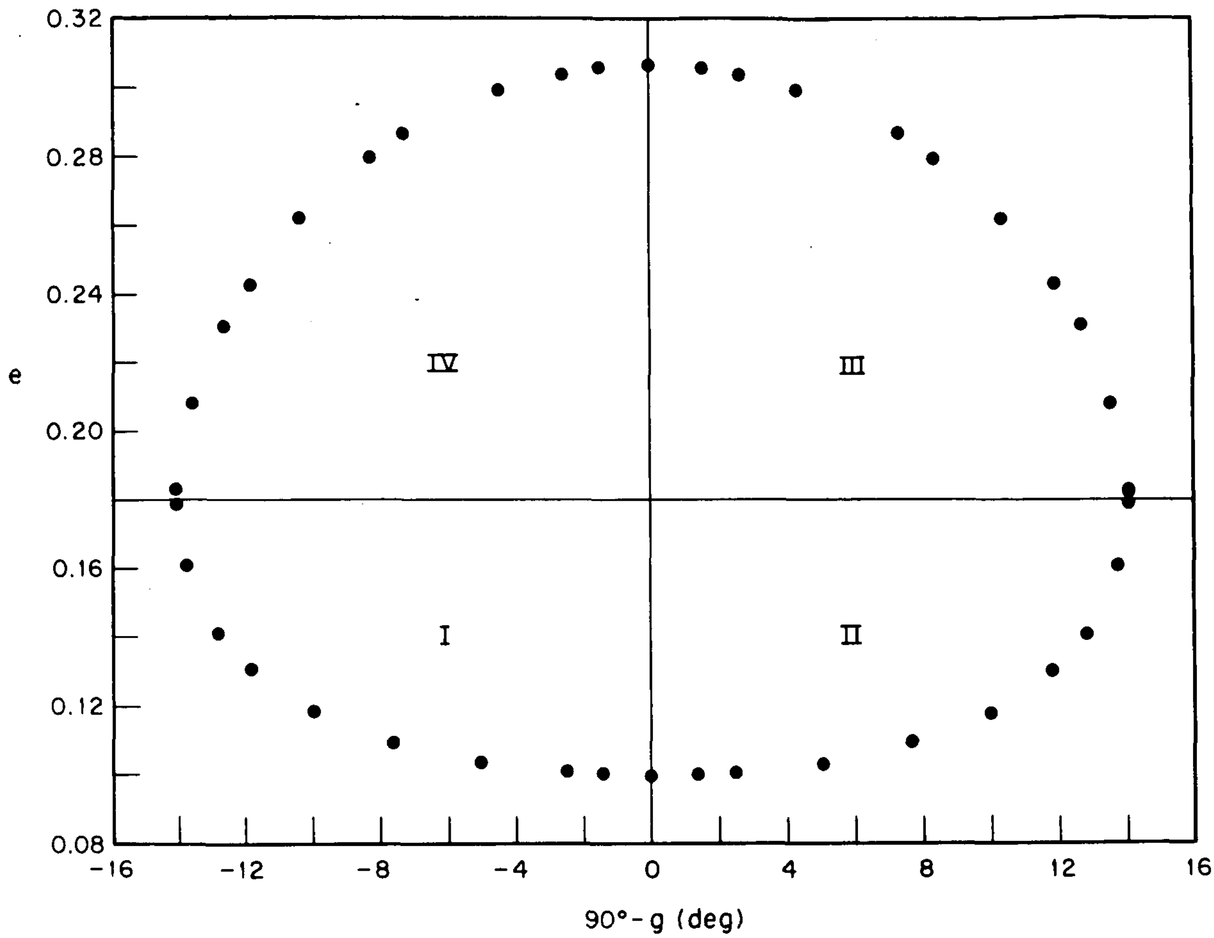

Figure 2–$\dot{e}$ vs. 90° – g.

consisting of quadrants I and II differing a little from the branch contained in quadrants III and IV.

EXPANSIONS IN PHASE SPACE

The results illustrated by Figure 3 may be utilized to find analytic expressions for the cycle, and subsequently to express the elements of the lunar satellite as functions of time.

It is convenient to define the non-dimensional variables u , w_1, and w_2 by

$$u = \frac{90^\circ - g}{14^\circ.03519}, \quad w_1 = \frac{\dot{g}}{2.9705764 \times 10^{-7}}, \quad w_2 = \frac{\dot{g}}{3.1289033 \times 10^{-7}},$$

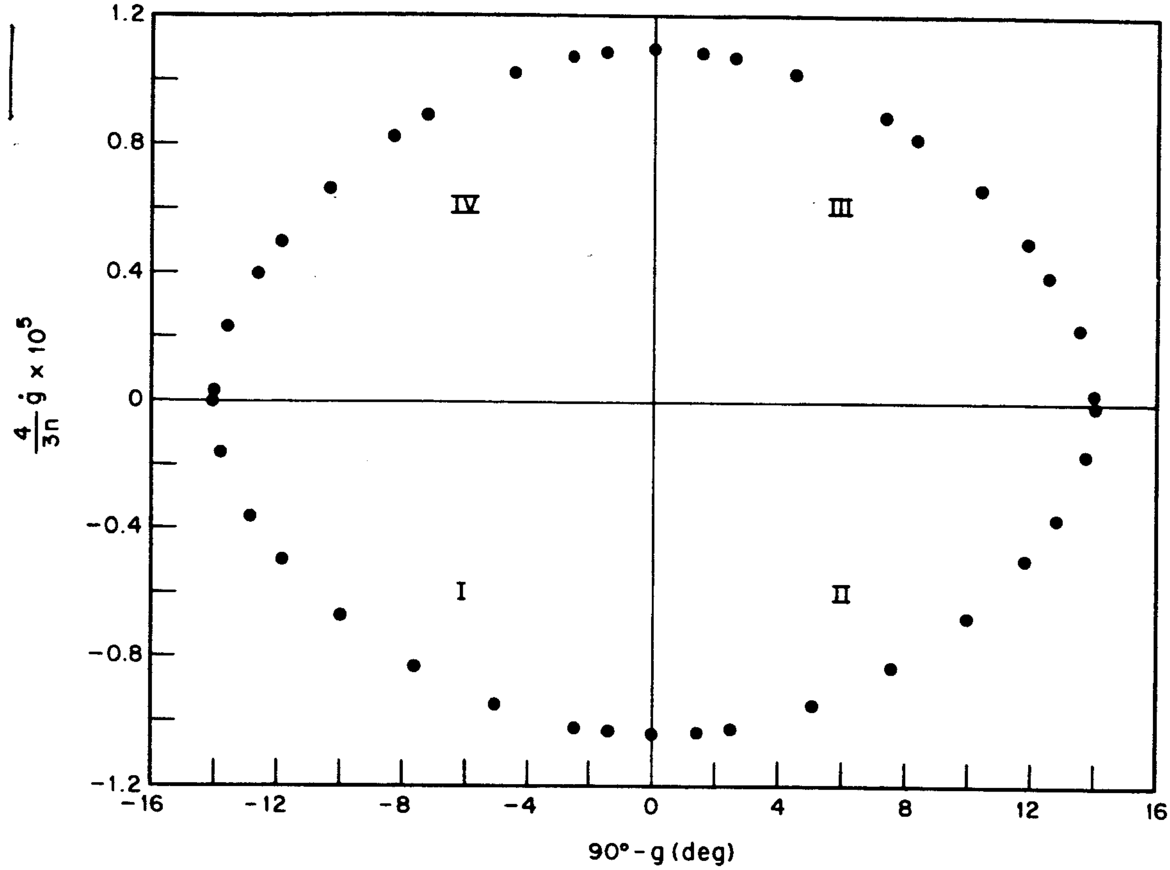

Figure 3—$\frac{4}{3n}\dot{g} \times 10^5$ vs. 90° − g

where g is in degrees and $\dot{g}$ is in canonical units. Values for u, w_1, and w_2 appear in Table 1. The angle 14°03519 is the variation of g about 90°, while 2.9705764×10^{-7} and $-3.1289033 \times 10^{-7}$ are, respectively, the maximum and minimum values of $\dot{g}$. Using the fact that the plot in Figure 3 consists of two branches, the following expansions were derived by least squares from the values in Table 1:

$$u = -\left\{\sqrt{1 - w_1^2} - .22131760\, w_1^2 + .20999084\, w_1^4\right\} \text{ for quadrant I}$$

$$u = \left\{\sqrt{1 - w_1^2} - .22131760\, w_1^2 + .20999084\, w_1^4\right\} \text{ for quadrant II}$$

$$u = \left\{\sqrt{1 - w_2^2} - .26127200\, w_2^2 + .24764299\, w_2^4\right\} \text{ for quadrant III}$$

$$u = -\left\{\sqrt{1 - w_2^2} - .26127200\, w_2^2 + .24764299\, w_2^4\right\} \text{ for quadrant IV} \qquad (11)$$

The expansions in Equations (11) were obtained by means of a desk calculator, and are therefore carried out to only a limited number of terms. However, they do represent the cycle within an accuracy of about 1%.

The next step in the process is to utilize Equations (11) to find the periodicities of the system, as well as expansions in terms of time.

EXPANSIONS OF $\dot{g}$, g, AND e IN TERMS OF TIME

The differential equation for the time is

$$dt = \frac{dg}{\dot{g}} ; \tag{12}$$

integration is performed over the cycle of Figure 3, with the aid of Equations (11).

Since there are essentially two different expansions in Equations (11) (one for each branch), we can determine two different periodicities in g and $\dot{g}$. Thus, we define

$$T_1 = \int_{c_1} \frac{dg}{\dot{g}} , \tag{13}$$

where the integration is performed over quadrant I or II, and

$$T_2 = \int_{c_2} \frac{dg}{\dot{g}} , \tag{14}$$

where the integration is performed over quadrant III or IV. Then, by substituting Equations (11) into Equations (13) and (14), we find that

$$T_1 = .82462207 \times 10^6 \left[\sin^{-1} w_1 + .44263520\, w_1 - .27998779\, w_1^3 \right]_{-1}^{0}$$

$$= 1.4294360 \times 10^6 \text{ canonical time units}$$

$$T_2 = .78289502 \times 10^6 \left[\sin^{-1} w_2 + .55254400\, w_2 - .33019065\, w_2^3\right]_0^1$$

$$= 1.3803611 \times 10^6 \text{ canonical time units.}$$

It follows that the period taken over the entire cycle of Figure 3 is

$$T = 2\,(T_1 + T_2).$$

Equations (11) and (12) can be utilized to obtain the time histories of g and $\dot{g}$. Thus, we find that

$$\begin{aligned}\bar{g}_1 &= \frac{\pi}{T_1} \times .82462207 \times 10^6 \left[\sin^{-1} w_1 + .44263520 w_1 - .27998779 w_1^3\right] + \text{const.} \\ &= \frac{\pi}{T_1}(t - t_0) + \text{const.}\end{aligned} \tag{15}$$

represents w_1 as a function of time in quadrant I, and $-\bar{g}_1$ represents w_1 in quadrant II. Similarly,

$$\begin{aligned}\bar{g}_2 &= \frac{\pi}{T_2} \times .78289502 \times 10^6 \left[\sin^{-1} w_2 + .52254400 w_2 - .33019065 w_2^3\right] + \text{const.} \\ &= \frac{\pi}{T_2}(t - t_0) + \text{const.}\end{aligned} \tag{16}$$

represents w_2 in quadrant III, and $-\bar{g}_2$ represents w_2 in quadrant IV.

Now, let us choose $t - t_0$ to be zero at the time of the initial conditions (i.e., the beginning of quadrant II); in addition, let $\bar{g}_1$ and $\bar{g}_2$ increase with time with mean motions π/T_1 and π/T_2, respectively. Further, let $\bar{g}_1 = 90°$ at $t - t_0 = 0$. Values of $\bar{g}_1$, $\bar{g}_2$, and $t - t_0$ appear in Table II.

Figures 4 and 5 are plots of u and e, respectively, vs. $t - t_0$. Least squares analyses yield the following results:

$$u_1 = -1.02099718 \cos \bar{g}_1 + .01610990 \cos 3\bar{g}_1 + .00649749 \cos 5\bar{g}_1 \tag{17}$$

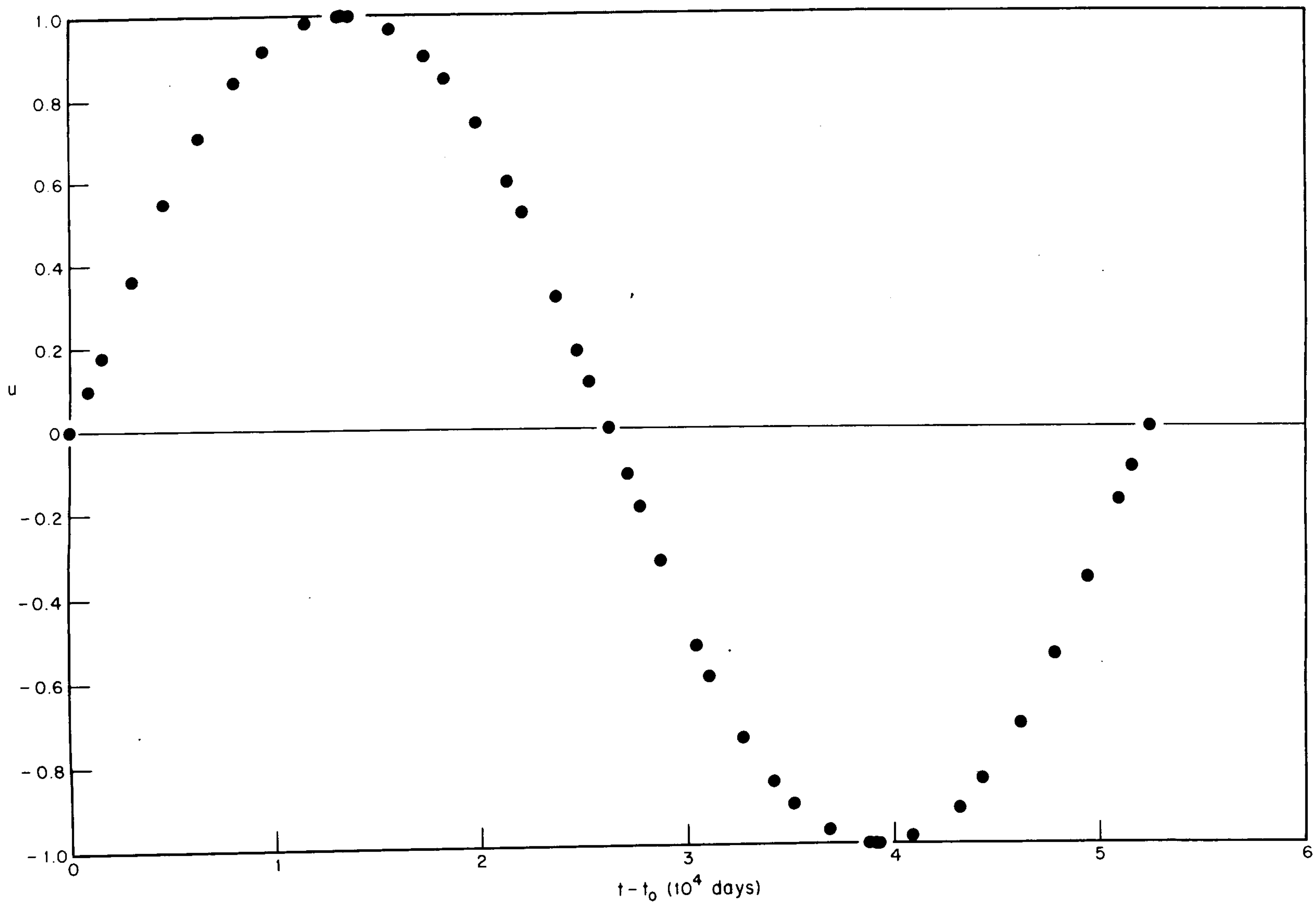

Figure 4—u vs. $t - t_0$.

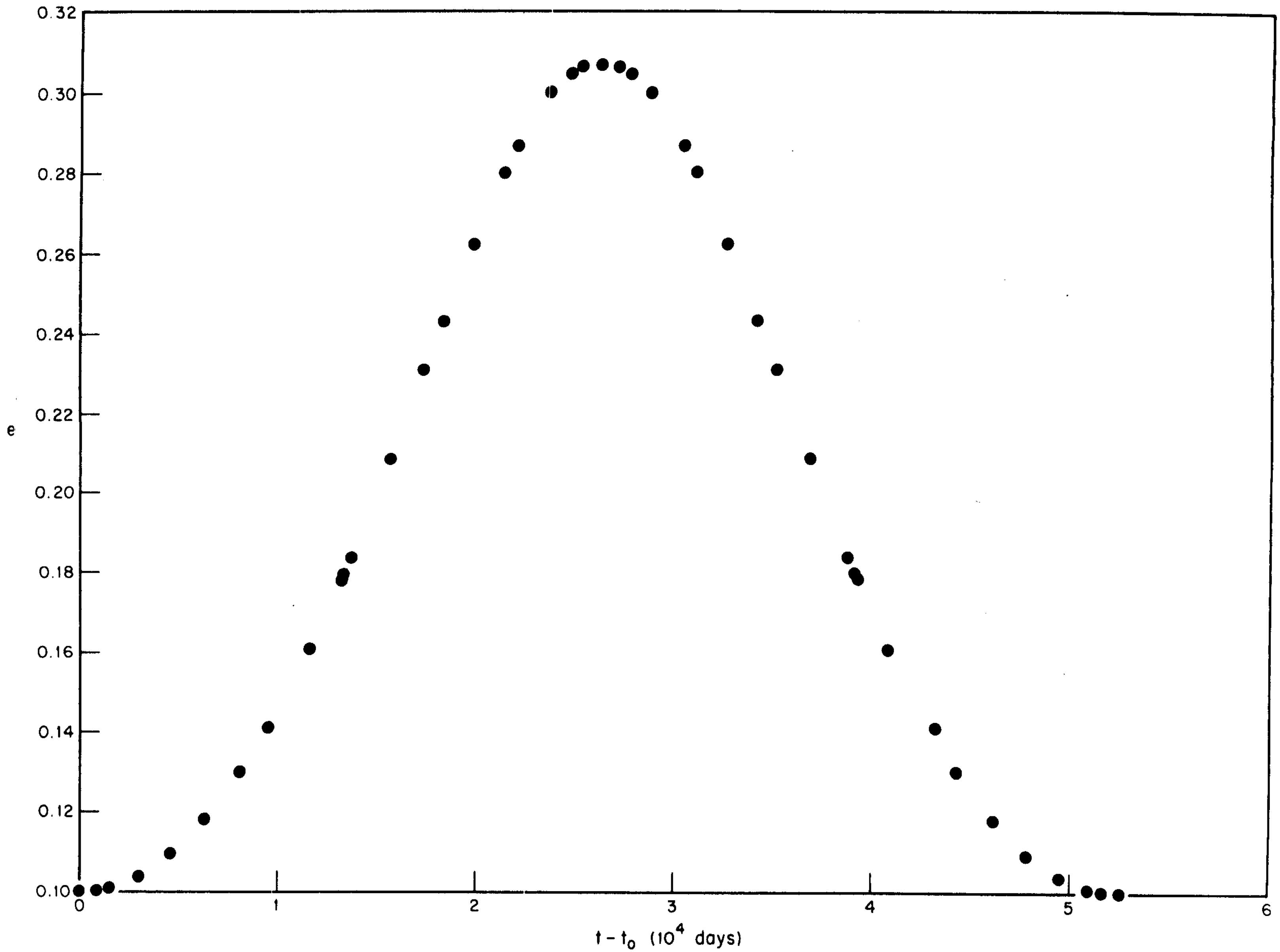

Figure 5—e vs. $t - t_0$.

$$e = .17962767 - .07962767\,(1.03054138 \sin \bar{g}_1 + .03349887 \sin 3\,\bar{g}_1$$

$$+ .00518834 \sin 5\,\bar{g}_1) \tag{18}$$

for quadrants I and II, and

$$u_2 = -1.02331813 \cos \bar{g}_2 + .01975752 \cos 3\,\bar{g}_2 + .00608506 \cos 5\,\bar{g}_2 \tag{19}$$

$$e = .17962767 + .12731988\,(-.95873486 \sin \bar{g}_2 + .04685613 \sin 3\,\bar{g}_2$$

$$+ .00286600 \sin 5\,\bar{g}_2) \tag{20}$$

for quadrants III and IV. Equations (17), (18), (19), and (20) represent the data to within about 2%.

ℓ AND h

Equations for the motions of the mean anomaly ℓ and the Delaunay variable h may be obtained from References 5 and 6 — they are as follows:

$$\dot{\ell} = n\left\{1 - \frac{1}{2}\frac{\alpha_1}{q}\left[10 + 3\nu^2 - 6\eta^2 - \frac{15\,(2-\eta^2)\,(\eta^2-\nu^2)}{\eta^2}\sin^2 g\right] - \frac{3}{4}\alpha_2\,\frac{\eta^2 - 3\nu^2}{\eta^5}\right\} \tag{21}$$

$$\dot{h} = -n_{☾}^{*} + n\nu\left\{-\frac{3}{4}\frac{\alpha_1}{q}\left[2 + \frac{10\,(1-\eta^2)}{\eta^2}\sin^2 g\right] - \frac{3}{2}\,\frac{\alpha_2}{\eta^5}\right\}. \tag{22}$$

(See Appendix A for definitions of constants and variables.)

Values for $\Delta\dot{\ell}/n \times 10^5$ and $\Delta\dot{h}/n\nu \times 10^5$ appear in Table III ($\Delta\dot{\ell} = \dot{\ell} - n$, and $\Delta\dot{h} = \dot{h} + n_{☾}^{*}$). In addition, Figures 6 and 7 show plots of $\Delta\dot{\ell}/n \times 10^5$ and $\Delta\dot{h}/n\nu \times 10^5$, respectively, vs. time. These can be integrated numerically to find the perturbations in ℓ and h.

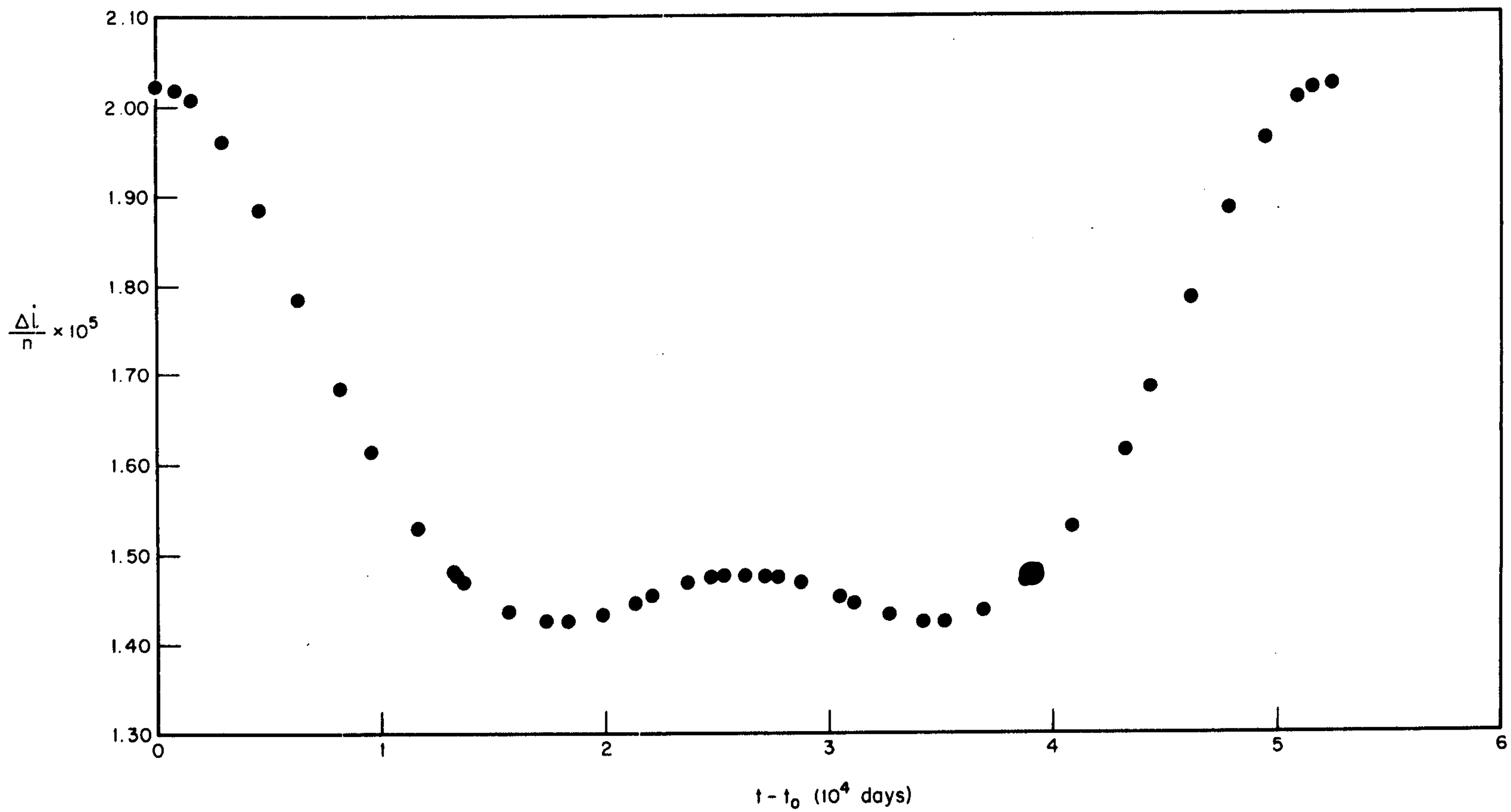

Figure 6—$\frac{\Delta \dot{\ell}}{n} \times 10^5$ vs. $t - t_0$.

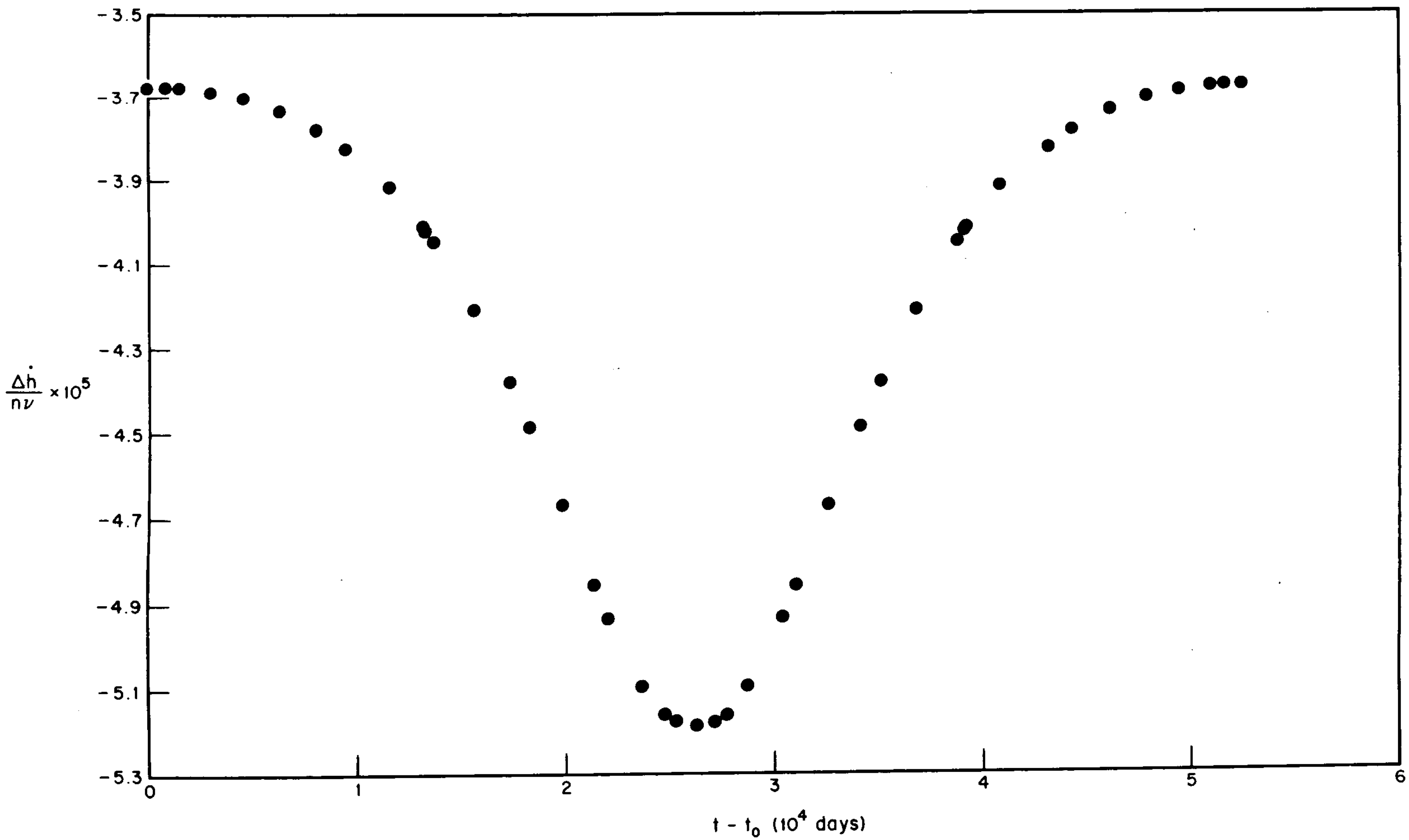

Figure 7–$\frac{\Delta \dot{h}}{n\nu} \times 10^5$ vs. $t - t_0$.

CONCLUSIONS

The type of motion known as libration is possible from lunar satellites — by examining the energy equation and the equation of motion of the argument of pericenter, initial conditions leading to libration can be found.

The methods of non-linear mechanics are useful for investigating the details of the motion. The procedure chosen for the example in this report may be adapted to electronic machine calculations.

The results presented here differ from those obtained through the use of usual perturbation theories in the following ways:

1. The variation in e is much larger than the order of magnitude of the disturbing function, which is about 10^{-5}.

2. The variation in the argument of pericenter can be explained without a secular term.

Finally, the fact that substantial changes may occur in the eccentricity of a lunar satellite in libratory motion should be taken into account in the planning of various types of orbits.

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Appendix A

SYMBOLS

a = semi-major axis of satellite's orbit

e = eccentricity of satellite's orbit

i = inclination of satellite's orbital plane to moon's equatorial plane

ℓ = mean anomaly of satellite

g = argument of pericenter of satellite's orbit

$h = \Omega - \lambda_{\oplus}$

Ω = longitude of ascending node of satellite's orbit

$\lambda_{\oplus}$ = mean longitude of earth, measured on moon's equator

n = mean motion of satellite

$\mu = n^2 a^3$

$L = \sqrt{\mu a}$

$\nu = \sqrt{1-e^2} \cos i$

$\eta = \sqrt{1-e^2}$

$a_{☾}$ = semi-major axis of moon's orbit $\simeq$ 384,400 Km.

$R_{☾}$ = mean radius of moon $\simeq$ 1738 Km.

J_2 = second zonal harmonic coefficient of moon $\simeq 2.41 \times 10^{-4}$

$n_{☾}^{*}$ = mean motion of $\lambda_{\oplus}$

$$q = 1 + \frac{\text{mass of moon}}{\text{mass of earth}}$$

$$\alpha_1 = \frac{1}{2}\left(\frac{a}{a_{☾}}\right)^3$$

$$\alpha_2 = R_{☾}^2 J_2/a^2$$

C = energy integral constant

Table I

Quadrants I and II

	90°-g (deg.)	e	$\frac{4}{3n}\dot{g} \times 10^5$	u	w_1
	0	.10000000	-1.0396376	0	-1
	∓ 1.40225	.10027263	-1.0328040	∓.09990958	-.99342694
	∓ 2.48788	.10086620	-1.0180892	∓.17726016	-.97927316
	∓ 5.08020	.10378266	- .94899765	∓.36196161	-.91281582
II	∓ 7.66137	.10938007	- .82966344	∓.54586864	-.79803139
	∓ 9.98252	.11811435	- .67139774	∓.71124937	-.64579979
	∓11.83747	.13010669	- .49435338	∓.84341359	-.47550549
I	∓12.85549	.14106736	- .36056622	∓.91594699	-.34681914
	∓13.79363	.16072025	- .16220364	∓.98278898	-.15601941
	∓14.03390	.17816846	- .01190701	∓.99990809	-.01145304
	∓14.03519	.17962767	0	∓1	0

Quadrants III and IV

	90°-g (deg.)	e	$\frac{4}{3n}\dot{g} \times 10^5$	u	w_2
	±14.03519	.17962767	0	±1	0
	±14.02577	.18360556	.03209762	±.99932883	.02931159
	±13.57598	.20860489	.22663012	±.96728153	.20695896
	±12.65501	.23080511	.39860814	±.90166289	.36400954
III	±11.92278	.24310492	.49708257	±.84949189	.45393654
	±10.39403	.26224988	.65845333	±.74056924	.60130056
	± 8.36212	.28000000	.82007223	±.59579671	.74889117
IV	± 7.33139	.28676820	.88544486	±.52235733	.80858956
	± 4.47007	.29980660	1.0181686	±.31849017	.92979307
	± 2.61962	.30453859	1.0687611	±.18664656	.97599422
	± 1.56060	.30609802	1.0857364	±.11119194	.99149609
	0	.30694755	1.0950486	0	1

Table II

	$t - t_0$ (10^4 days)		$\bar{g}_1$ (deg.)	
	Quadrant II	Quadrant I	Quadrant II	Quadrant I
	0	5.2477884	90	90
	.0863580	5.1614304	95.822496	84.177504
	.1509922	5.0967962	100.180309	79.819691
II	.3020400	4.9457484	110.364367	69.635633
	.4608704	4.7869180	121.073147	58.926853
	.6321681	4.6156203	132.622511	47.377489
I	.8143666	4.4334218	144.906837	35.093163
	.9529006	4.3218878	154.247180	25.752820
	1.1618342	4.0859342	168.334056	11.665944
	1.3221378	3.9256506	179.142166	0.857834
	1.3348611	3.9129273	180	0

	$t - t_0$ (10^4 days)		$\bar{g}_2$ (deg.)	
	Quadrant III	Quadrant IV	Quadrant III	Quadrant IV
	1.3348611	3.9129273	180	0
	1.3674856	3.8803028	182.277837	357.722163
	1.5641943	3.6835941	196.011995	343.988005
III	1.7346646	3.5131238	207.914188	333.085812
	1.8301755	3.4176129	214.582740	325.417260
	1.9837445	3.2640439	225.304893	314.695107
IV	2.1383611	3.1094273	236.100188	303.899812
	2.2046564	3.0431320	240.728914	299.271086
	2.3688471	2.8789413	252.192670	287.807330
	2.4711777	2.7766107	259.337369	280.662631
	2.5313346	2.7164538	263.537508	276.462492
	2.6238942	2.6238942	270	270

Table III

Quadrants I and II

	$\frac{\Delta \dot{\ell}}{n} \times 10^5$	$\frac{\Delta \dot{h}}{n\nu} \times 10^5$
	2.0223381	-3.6754823
	2.0177568	-3.6762854
	2.0079104	-3.6780420
II	1.9620501	-3.6868374
	1.8845147	-3.7044884
	1.7859026	-3.7340862
	1.6835939	-3.7789088
I	1.6141463	-3.8242497
	1.5284436	-3.9165212
	1.4805491	-4.0109717
	1.4774378	-4.0194326

Quadrants III and IV

	1.4774378	-4.0194366
	1.4695560	-4.0429523
	1.4368206	-4.2067104
III	1.4264079	-4.3771699
	1.4264217	-4.4828268
	1.4332899	-4.6648510
	1.4462603	-4.8546807
IV	1.4527512	-4.9329060
	1.4675579	-5.0934689
	1.4736671	-5.1551381
	1.4757658	-5.1758757
	1.4769324	-5.1872620